Linearne diferencijalne jednačine II reda sa konstantnim koeficijentima

Opšti oblik $y''+py'+qy=f(x)$, p,q - const.
$(*)$ $f(x)\in C$

$f(x)=0 \Rightarrow$ homogeni dio posmatrane j-ne

Kako se ona rj:

$$y''+py'+qy=0 \xrightarrow{\text{karakt. j-n}} \text{odgovarajuća kv. j-na je } k^2+pk+q=0$$

① $k_1\neq k_2,\ k_1,k_2\in\mathbb{R} \Rightarrow y_{hom}=C_1e^{k_1x}+C_2e^{k_2x}$

② $k_1=k_2\in\mathbb{R} \Rightarrow y_{hom}=C_1e^{k_1x}+C_2xe^{k_1x}$

③ $k_1=\overline{k_2}\in\mathbb{C} \Rightarrow y_{hom}=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

$k_1=\alpha+i\beta$
$k_2=\alpha-i\beta$

Opšte rješenje j-ne $(*)$ je $y=y_{hom}+y_p$.

y_p partikularno rj tražimo zavisno od $f(x)$

① ako je $f(x)=e^{\lambda x}P_n(x)$; $P_n(x)$ polinom n-tog stepena tada je

$$y_p=\begin{cases} e^{\lambda x}Q_n(x), & \lambda \text{ nije rj. karakt. j-ne}\\ xe^{\lambda x}Q_n(x), & \lambda \text{ jednostruko rj karakt. j-ne}\\ x^2e^{\lambda x}Q_n(x), & \lambda \text{ dvostruko rj karakt. j-ne}\end{cases}$$

Q_s, R_s - polinomi st. $s=\max\{m,n\}$

$Q_n(x)$ - polinom n-tog st. čije koef određujemo iz $y_p''+py_p'+qy_p=f(x)$

② $f(x)=e^{\alpha x}(P_n(x)\cos\beta x+T_m(x)\sin\beta x)$, P_n - polinom n-tog, T_m - m-tog st.

$$y_p=\begin{cases} e^{\alpha x}(Q_s(x)\cos\beta x+R_s(x)\sin\beta x), & \text{ako } \alpha+i\beta \text{ nije rj. karakt. jed.}\\ xe^{\alpha x}(Q_s(x)\cos\beta x+R_s(x)\sin\beta x), & \text{-"- -"- jeste rj -"-}\end{cases}$$

③ ako je $f = f_1 + f_2$, rješavamo $y'' + py' + qy = f_1 \wedge y'' + py' + qy = f_2$

$$\underset{y_{p_1} = y_1}{\downarrow} \qquad y_{p_2} = y_2$$

$\Rightarrow y_p = y_1 + y_2$

1) Riješiti dif. j-nu: $y'' - y = x^2 + 1$

Rješavamo homogeni dio: $y'' - y = 0$

$k^2 - 1 = 0$

$k^2 = 1 \Rightarrow \underline{k = \pm 1}$

$y_h = C_1 e^x + C_2 e^{-x}$.

Tražimo y_p? $y'' - y = x^2 + 1$, $f(x) = x^2 + 1 \Rightarrow$ polinom II st.

$\Rightarrow x^2 + 1 = e^{0 \cdot x}(x^2 + 1) = e^{0 \cdot x} \cdot P_2(x)$, a 0 nije rj karakt. j-ne $\Rightarrow$

$y_p = ax^2 + bx + c$, $a, b, c = ?$ $y_p'' - y_p = x^2 + 1$

$$\cancel{\begin{matrix} 2ax - ax^2 - bx - c = x^2 + 1 \\ \left.\begin{matrix} -a = 1 \\ 2a - b = 0 \\ -c = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} \boxed{a = -1} \\ \boxed{b = -2} \\ \boxed{c = -1} \end{matrix} \end{matrix}} \qquad y_p = -x^2 - 2x - 1$$

$y_p' = 2ax + b \Rightarrow y_p'' = 2a$

$$2a - ax^2 - bx - c = x^2 + 1 \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ c = -3 \end{cases} \qquad \underline{y_p = -x^2 - 3}$$

$\underline{y = C_1 e^x + C_2 e^{-x} - x^2 - 3}$

② $y'' + y = \sin x$

$y'' + y = 0$

$k^2 + 1 = 0$

$k^2 = -1 \Rightarrow k = \pm\sqrt{-1}$

$\underbrace{k_1 = i}, \underbrace{k_2 = -i}$

$k_1 = \underset{\alpha}{0} + \underset{\beta}{1} \cdot i, \quad k_2 = \underset{\alpha}{0} + \underset{\beta}{(-1)} i$

$x_h = e^{0x}(C_1 \cos x + C_2 \sin x) =$

$= C_1 \cos x + C_2 \sin x$

$y_p = ?$ $\quad f(x) = \sin x = e^{0 \cdot x}(0 \cdot \cos\beta x + 1 \cdot \sin x)$

$m = 0, \quad \beta = 1$

i je rj. karakt. j-ne $\Rightarrow$ $y_p = x e^{0x}(R_0(x) \cos x + S_0(x) \sin x) =$

$= x(a \cos x + b \sin x)$

$y_p'' + y_p = \sin x, \quad y_p' = a\cos x + b \sin x + x(-a \sin x + b \cos x)$

$y_p'' = -a \sin x + b \cos x - a \sin x + b \cos x + x(-a\cos x - b \sin x)$

$-a\sin x + b\cos x - a\sin x + b\cos x - \cancel{ax\cos x} - \cancel{xb\sin x} + \cancel{ax\cos x} + \cancel{xb\sin x} = \sin x$

$-2a \sin x + 2b\cos x = \sin x$

$-2a = 1 \Rightarrow \underbrace{a = -\frac{1}{2}}, \quad \underbrace{b = 0}, \quad \underbrace{y_p = -\frac{1}{2} x \cos x}$

$\underbrace{y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x}$

3) $y'' - 2y' + y = e^x + \sin^2 x$

$y'' - 2y' + y = 0$

$k^2 - 2k + 1 = 0 \Rightarrow (k-1)^2 = 0$

$\boxed{k = 1}$

$\underbrace{x_h = C_1 e^x + C_2 x e^x}$

Tražimo $y_p = ?$

$$f(x) = e^x + \sin^2 x = \cancel{f_1(x) + f_2(x)} = e^x + \frac{1-\cos 2x}{2} = e^x + \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= f_1 + f_2 + f_3$$

$$y_p = y_1 + y_2 + y_3 \quad , \boxed{k=1}$$

$y'' - 2y' + y = e^x = e^{1\cdot x} P_0(x)$, 1 jeste rj. kar. j-ne i to dvostruko

$$y_1 = x^2 e^x Q_0(x) = x^2 e^x \cdot a$$

$$y_p' = 2xe^x a + x^2 e^x a =$$

$$y_p'' = 2a(e^x + xe^x) + a(2xe^x + x^2 e^x)$$

$$2xe^x a + x^2 e^x a - 4a(e^x + xe^x) - 2a(2xe^x + x^2 e^x) + ax^2 e^x = e^x$$

$$2xe^x a - 4ae^x - 4axe^x - 4axe^x = e^x$$

$$-6xe^x a - 4ae^x = e^x$$

$$2ae^x(-2-3x) = e^x$$

$$2a(-2-3x) = 1$$

$$y'' - 2y' + y = \frac{1}{2} = e^{0\cdot x} P_0(x)$$

$$\boxed{y_2 = b}$$

$(0+2i)$ nije rj.

$$y'' - 2y' + y = -\frac{1}{2}\cos 2x = e^{0x}\left(P_0(x)\cdot\cos 2x + 0\cdot\sin\beta x\right) \Rightarrow$$

$$y_3 = \left(Q_0(x)\cos 2x + R_0(x)\sin 2x\right) = c\cos 2x + d\sin 2x$$

$$y_p = ax^2e^x + b + c\cos 2x + d\sin 2x$$

$$y_p' = a(2xe^x + x^2e^x) - 2c\sin 2x + 2d\cos 2x$$

$$y_p'' = a(2e^x + 2xe^x + 2xe^x + x^2e^x) - 4c\cos 2x - 4d\sin 2x$$

$$a(2e^x + \cancel{4xe^x} + \cancel{x^2e^x}) - 4c\cos 2x - 4d\sin 2x - \cancel{4axe^x} - \cancel{2ax^2e^x} + 4c\sin 2x$$

$$- 4d\cos 2x + \cancel{ax^2e^x} + b + c\cos 2x + d\sin 2x = e^x + \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$2ae^x - \cos 2x(4c + 4d - c) - \sin 2x(-4a - 4c - d) + b =$$

$2a = 1$

$\boxed{a = \frac{1}{2}}$

$3c + 4d = +\frac{1}{2} \ /\cdot 2$

$-4a - 4c - d = 0$

$-2\cancel{a} - 4c - d = 0$

$4c + d = -2$

$\boxed{b = \frac{1}{2}}$

$$\begin{cases} 6c + 8d = -1 \\ 4c + d = -2 \Rightarrow d = -2 - 4c \end{cases}$$

$6c - 16 - 32c = -1$

$-26c = 15$

$\boxed{c = -\frac{15}{26}}$

$d = -2 - \cancel{4}^{2} \cdot \frac{-15}{\cancel{26}_{13}}$

$d = -2 + \frac{30}{13} = \frac{4}{13}$

$\boxed{3c + 4d = 1}$

$d = -2 - 4c$

$3c - 8 - 16c = 1$

$-13c = 9$

$c = -\frac{9}{13}$

$$y'' - 3y' + 2y = \sin x$$

$y_h = ?$ $\quad y'' - 3y' + 2y = 0$

$$k^2 - 3k + 2 = 0$$

$$k_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$$

$$k_1 = 2, \quad k_2 = -1$$

$$\rightarrow y_h = C_1 e^{2x} + C_2 e^{-x}$$

$y_p = ?$ $\quad \cancel{\sin}\ f(x) = \sin x = e^{0x}(1 \sin x + 0 \cdot \cos x)$

nije rj. homog. dijela

polinom nultog st.

$$\Rightarrow y_p = A \sin x + B \cos x$$

$$y_p' = A \cos x - B \sin x$$

$$y_p'' = -A \sin x - B \cos x$$

$$-A \sin x - B \cos x - 3(A \cos x - B \sin x) + 2A \sin x + 2B \cos x = \sin x$$

$$\sin x(-A + 3B + 2A) + \cos x(-B - 3A + 2B) = \sin x$$

$$A + 3B = 1 \qquad A + 3 \cdot 3A = 1$$

$$B - 3A = 0 \Rightarrow \boxed{B = 3A} \qquad 10A = 1 \qquad \boxed{A = \frac{1}{10}} \qquad \boxed{B = \frac{3}{10}}$$

$$y_p = \frac{1}{10} \sin x + \frac{3}{10} \cos x$$

$$\boxed{y = y_p + y_h}$$

$$y'' + y' - 2y = (x^2 - 1)e^{2x}$$

$$y'' + y' - 2y = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 1$$

$$y_h = C_1 e^{x} + C_2 e^{-2x}$$

$y_p = ?$

$$f(x) = (x^2-1)e^{2x} = e^{2x}(x^2-1)$$

2 nije rj. kar. j-ne

$$y_p = \underline{e^{2x}}(ax^2 + bx + c)$$

$$y_p' = e^{2x} \cdot 2(ax^2 + bx + c) + e^{2x}(2ax + b)$$

$$= e^{2x}(2ax^2 + 2bx + 2ax + 2c + b)$$

$$y_p'' = e^{2x} \, 2(2ax^2 + 2bx + 2ax + 2c + b) +$$

$$+ e^{2x}(4ax + 2b + 2a) =$$

$$= 2e^{2x}(2ax^2 + 2bx + 2ax + 2c + b + 2ax + b + a) =$$

$$= 2e^{2x}(2ax^2 + 2bx + 4ax + 2b + a + 2c)$$

$$\cancel{2e^{2x}} e^{2x}(4ax^2 + 4bx + 8ax + 4b + 2a + 4c + \cancel{2ax^2} + \cancel{2bx} + 2ax + 2c + b -$$

$$- \cancel{2ax^2} - \cancel{2bx} - 2c) = x^2 e^{2x} - e^{2x}$$

$$\left.\begin{aligned} 4a &= 1 \\ 10a + 4b &= 0 \\ 5b + 2a + 4c &= -1 \end{aligned}\right\}$$

$$a = \frac{1}{4}$$

$$b = \frac{-10a}{4} = \frac{-10 \cdot \frac{1}{4}}{4} = \frac{-5}{8}$$

$$5 \cdot \left(-\frac{5}{8}\right) + 2 \cdot \frac{1}{4} + 4c = -1$$

$$c = \frac{13}{32}$$

$$y_p = e^{2x}\left(\frac{1}{4}x^2 - \frac{5}{8}x + \frac{13}{32}\right) \qquad y = y_h + y_p$$

3.5. ZADACI

1. Riješiti diferencijalnu jednačinu $x^2 + y^2 - 2xyy' = 0$. **Rješenje:** $x^2 - y^2 = Cx$.

2. Riješiti diferencijalnu jednačinu $\sqrt{y^2+1}dx = xydy$. **Rješenje:** $x = ce^{\sqrt{y^2+1}}$.

3. Riješiti diferencijalnu jednačinu $(x^2 - yx^2)y' + y^2 + xy^2 = 0$. **Rješenje:** $y = ce^{-\frac{1}{x}-\frac{1}{y}}$.

4. Riješiti diferencijalnu jednačinu $xyy' = \sqrt{x^4 - y^4}$. **Rješenje:** $\arcsin\frac{y^2}{x^2} = x^2\ln\left|\sqrt{x^4+y^4} - y^2\right| + C$.

5. Riješiti diferencijalnu jednačinu $xy' - 4y - x^2\sqrt{y} = 0$. **Rješenje:** $y = x^4\left(\frac{1}{2}\ln|x| + C\right)^2$.

6. Riješiti diferencijalnu jednačinu $(xy^2 + x)dx + (y - x^2y)dy = 0$. **Rješenje:** $y^2 + 1 = c\cdot(1 - x^2)$.

7. Riješiti diferencijalnu jednačinu $xy(1 + xy^2)y' = 1$. **Rješenje:** $x = \frac{1}{Ce^{-\frac{y^2}{2}} - y^2 + 2}$

8. Riješiti diferencijalnu jednačinu $2x^2yy' + y^2 = 2$. **Rješenje:** $\frac{c}{2 - y^2} = e^{-\frac{1}{x}}$.

9. Naći krive kod kojih je odsječak tangente u ma kojoj tački između koordinatnih osa podijeljen tom tačkom na dva jednaka dijela. Zatim naći onu krivu koja prolazi kroz tačku $M(3,2)$. **Rješenje:** $xy = c, \quad xy = 6$.

10. Naći jednačinu krive koja prolazi kroz tačku $A(0,2)$ ako je površina krivolinijskog trapeza ograničenog lukom krive i koordinatnim osama dva puta veća od dužine odgovarajućeg luka krive.**Rješenje:** $2\ln\left|y + \sqrt{y^2 - 4}\right| = \pm x + \ln 4$

11. Riješiti diferencijalnu jednačinu $(y^2 - 2xy)dx + x^2dy = 0$. **Rješenje:** $x = \frac{cy}{x - y}$.

12. Riješiti diferencijalnu jednačinu $xy' - y = (x + y)\ln\frac{x+y}{x}$. **Rješenje:** $1 + \frac{y}{x} = e^{cx}$.

13. Riješiti diferencijalnu jednačinu $(xy + e^x)dx - xdy = 0$. **Rješenje:** $y = e^x(\ln x + c)$.

14. Riješiti diferencijalnu jednačinu $xy' + (x+1)y = 3x^2e^{-x}$ **Rješenje:** $y = \frac{e^{-x}}{x}(x^3 + c)$

15. Odrediti jednačinu krive koja prolazi kroz tačku $A(0,1)$ ako je površina trapeza